Linear Systems on Graphs

Controllability of Structured Networks

Chapter 2

Foundations

Graph of a Linear System · Walks and the Controllability Matrix · Consensus

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Chapter 2 Contents

The LTI State-Space Model: State as Memory

CONCEPT State = Memory

- Every system here: **linear, time-invariant (LTI), lumped** $\rightarrow$ memory = a finite **state** vector $x(t) \in \mathbb{R}^n$

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du \qquad \text{(almost always } D = 0\text{)}$$

- $x \in \mathbb{R}^n$ state, $u \in \mathbb{R}^m$ input, $y \in \mathbb{R}^q$ output. A, B, C, D **constant**
- x_{init} with u on $t \geq 0$ $\rightarrow$ the **entire future**. Nothing earlier needed
- n **states** $\rightarrow$ n **state nodes** (next frame)

- Two descriptions $\rightarrow$ **internal** (the state, this course) and **external** (input $\rightarrow$ output, state eliminated $\rightarrow$ nothing to draw)

The state is the **smallest memory** that makes the future a function of the present state and the future input.

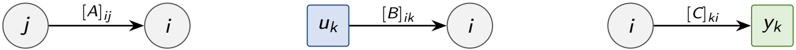
The Graph of a Linear System: One Node per State

CONCEPT Graph of a Linear System

$\dot{x} = Ax + Bu$, $y = Cx$ defines a **weighted directed graph** $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \mathcal{V}^S \cup \mathcal{V}^I \cup \mathcal{V}^O$ (one node per state, per input, per output) and

$$\begin{aligned}
(j, i) \in \mathcal{E} &\iff [A]_{ij} \neq 0 \\
(u_k, i) \in \mathcal{E} &\iff [B]_{ik} \neq 0 \\
(i, y_k) \in \mathcal{E} &\iff [C]_{ki} \neq 0
\end{aligned}$$

each edge carrying that entry as its **weight**.



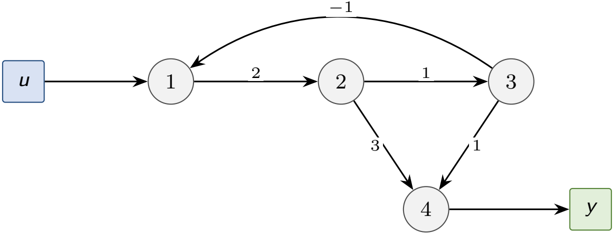
- **Direction**: $[A]_{ij} \neq 0 \;\rightarrow\;$ state j influences state $i \;\rightarrow\;$ edge $j \rightarrow i$
- **Column** index j influences, **row** index i is influenced
- $A =$ **weighted adjacency matrix**, transpose convention, fixed once
- Input edge into node $i \;\rightarrow\;$ node $i =$ **driver** (or **leader**) node

*Every linear system can now be **drawn*** $\rightarrow$ nodes = states, arrows = non-zero entries.

Matrix and Picture: A Four-Node Network

EXAMPLE Four-node network

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^\top$$



$1 \to 2 \to 3 \to 1$ returns to its start. $2 \to 4$ and $3 \to 4$ leave it, node 4 has no edge back

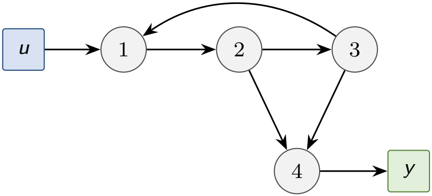
Sixteen entries in A, five arrows in the picture. **Which is more intuitive?**

Reading Every Entry: Five Edges, One Input, One Output

EXAMPLE Four-node network

- $[A]_{21} = 2 \;\rightarrow\;$ edge $1\rightarrow 2$
- $[A]_{32} = 1 \;\rightarrow\;$ edge $2\rightarrow 3$
- $[A]_{13} = -1 \;\rightarrow\;$ edge $3\rightarrow 1$
- $[A]_{42} = 3 \;\rightarrow\;$ edge $2\rightarrow 4$
- $[A]_{43} = 1 \;\rightarrow\;$ edge $3\rightarrow 4$

- $[B]_{11} = 1 \;\rightarrow\; u\rightarrow 1$, node 1 = **driver**
- $[C]_{14} = 1 \;\rightarrow\;$ edge $4\rightarrow y$



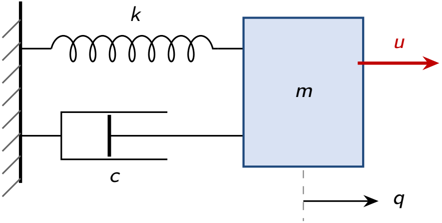
same picture, weights listed at left

- Matrix and picture: **identical information**. Ch. 1 opened with this picture and symbols on the edges $\rightarrow$ here it carries numbers

Row first = where the arrow **lands**. **Column** second = where it **starts**. $\rightarrow$ $[A]_{42} = 3$ draws the arrow $2\rightarrow 4$ of weight 3.

Mass, Spring and Damper: The Physical System

EXAMPLE Mass–spring–damper



q measured from equilibrium

$$m\ddot{q} = -kq - c\dot{q} + u$$

m mass, c damping, k stiffness, q position $\to$ the physical letters of this example, not the input and output dimensions m, q of $\dot{x} = Ax + Bu$, $y = Cx + Du$

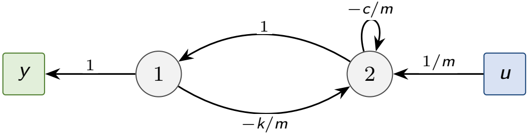
state $x = (q, \dot{q}) = (x_1, x_2)$, output $=$ position $\to$ $\dot{x}_1 = x_2$, $\quad \dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u$

Which two numbers are the memory? $\to$ position q and velocity $\dot{q}$, nothing else.

From Equation to Graph: Mass–Spring–Damper

EXAMPLE Mass–spring–damper

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$



- $[A]_{12} = 1 \;\rightarrow\;$ edge $2 \rightarrow 1$
- $[A]_{21} = -k/m \;\rightarrow\;$ edge $1 \rightarrow 2$
- $[A]_{22} = -c/m \;\rightarrow\;$ edge $2 \rightarrow 2$, a **self-loop** (back to its own node)
- $[B]_{21} = 1/m \;\rightarrow\; u \rightarrow 2$, **driver** node
- $[C]_{11} = 1 \;\rightarrow\;$ edge $1 \rightarrow y$

The **equation** is the model. The **picture** is the same model, one arrow per non-zero entry, one missing arrow per zero.

The One Move: Erase the Weights

CONCEPT Pattern Graph

- Every entry of A: a **structural zero** (no edge) or a **free non-zero** parameter a_{ij} (edge, value unknown)
- That matrix = **pattern matrix**. Its picture with the numbers rubbed off = **pattern graph**
- One choice of the free entries = a **realization** (Ch. 1)

- So far **edge weights** (the numbers) and **edge existence** travel together
- **Part II** keeps only **edge existence**
- **Walks** and the **controllability matrix** $\to$ **Part II**'s algebra, **edge existence** only
- **Which questions survive the erasure?** $\to$ the subject of **Part II**

"**Part II** will do exactly one thing to this picture: **forget the numbers on the edges** and keep only which edges exist."

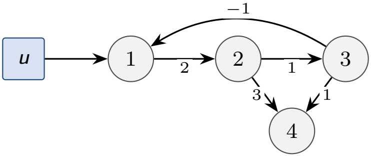
Chapter 2 Contents

Walk, Path, Cycle, Stem: Four Definitions

CONCEPT Four Definitions

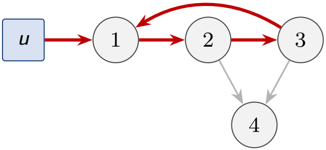
- **WALK** of length k $\rightarrow$ k consecutive directed edges, nodes may repeat. Empty sequence $=$ walk of length 0, a node to itself
- **PATH** $\rightarrow$ a walk visiting no node twice
- **CYCLE** $\rightarrow$ a path whose final edge returns to its start node
- **STEM** $\rightarrow$ a path originating at an input node

EXAMPLE Four-node network

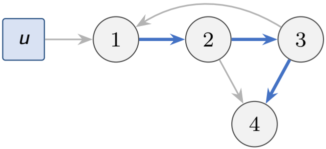


Four words, one picture. **Which of the four may visit a node twice?**

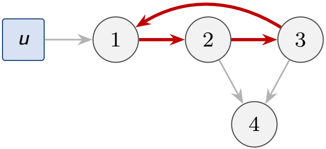
The Same Picture, Four Times: Walk, Path, Cycle, Stem



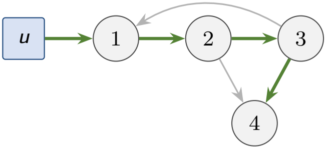
WALK of length 4
$u \to 1 \to 2 \to 3 \to 1$, node 1 twice



PATH of length 3
$1 \to 2 \to 3 \to 4$, no node twice



CYCLE of length 3
$1 \to 2 \to 3 \to 1$, back to its start

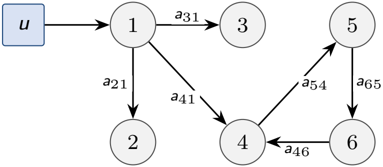


STEM out of the input node
$u \to 1 \to 2 \to 3 \to 4$, every state node

Walks and paths differ exactly at **cycles**. **Which of the four does A^k count?**

A Six-Node Network: One Driver, One Cycle

EXAMPLE **Network of the ECC 2024 paper, Fig. 1(b)**



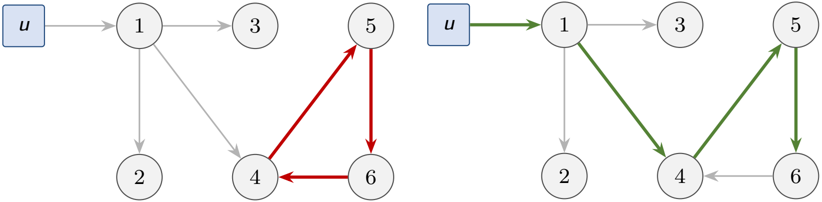
$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 & 0 \\ a_{31} & 0 & 0 & 0 & 0 & 0 \\ a_{41} & 0 & 0 & 0 & 0 & a_{46} \\ 0 & 0 & 0 & a_{54} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{65} & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Six nodes, six edges, one input at node 1 $\rightarrow$ driver node. Symbolic weights: edge $j \rightarrow i$ carries a_{ij}

Edge existence fixed, every edge weight still free. **What can the picture alone already tell us?**

Rows In, Columns Out: Reading the Six-Node Matrix

- Row i of A $\to$ the edges **entering** node i. Column j $\to$ the edges **leaving** node j
- Row 1 zero $\to$ nothing enters node 1. Columns 2 and 3 zero $\to$ nodes 2 and 3 have no edge out
- Two structures to read straight off the picture:



CYCLE $4 \to 5 \to 6 \to 4$

STEM $u \to 1 \to 4 \to 5 \to 6$, four state nodes

Six edges, one cycle, one stem. **Where can the input reach in exactly three steps?**

The Key Identity: Powers Count Walks

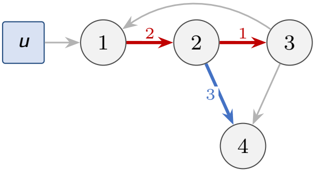
▶ **Walk enumeration, the key identity of Part II**

$$[A^k]_{lw} \;=\; \sum_{k\text{-step walks } w\to l} \text{(product of the edge weights along the walk)}$$

row l = where the walk **ends** · column w = where it **starts** · all weights 1 $\to$ a plain **count**

Induction step: $[A^{k+1}]_{lw} = \sum_j [A]_{lj}\,[A^k]_{jw}$ $\to$ a $(k+1)$-walk = a k-walk to some j, then the edge $j \to l$

EXAMPLE Two steps out of node 1, on the four-node network

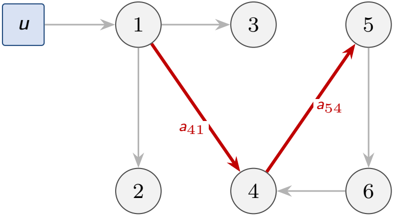


- $1 \to 2 \to 3$ $\to$ $[A^2]_{31} = 1 \cdot 2 = 2$
- $1 \to 2 \to 4$ $\to$ $[A^2]_{41} = 3 \cdot 2 = 6$
- No two-step walk into node 2 $\to$ $[A^2]_{21} = 0$ (node 1 has no edge into itself)

Every entry of A^k is a sum over walks. **So what does a zero entry mean?**

A^2 Out of the Driver: One Walk, One Entry

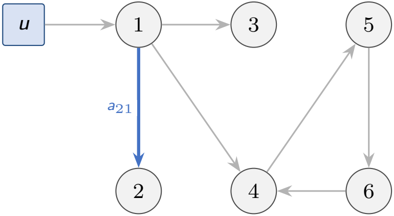
▶ Two steps out of the driver node



$1 \to 4 \to 5$

the only two-step walk out of the driver

$[A^2]_{51} = [A]_{54}[A]_{41} = a_{54}a_{41}$



node 2, reached in one step

never in two

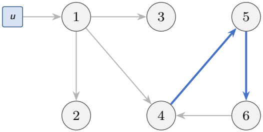
$[A^2]_{21} = 0$

- Node 2 has one edge in, from node 1, and node 1 has no self-loop $\to$ no walk of length 2 ends at node 2

Read the product right to left: $[A^2]_{51} = [A]_{54}[A]_{41}$, the walk $1 \to 4 \to 5$ composed edge by edge, **destination first**.

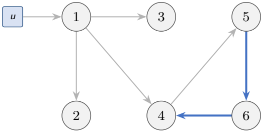
A^2 Inside the Cycle: Three More Entries

▶ The three two-step walks that stay inside the cycle



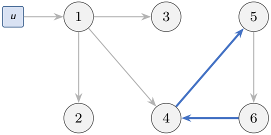
$4 \to 5 \to 6$

$[A^2]_{64} = a_{65}\, a_{54}$



$5 \to 6 \to 4$

$[A^2]_{45} = a_{46}\, a_{65}$



$6 \to 4 \to 5$

$[A^2]_{56} = a_{54}\, a_{46}$

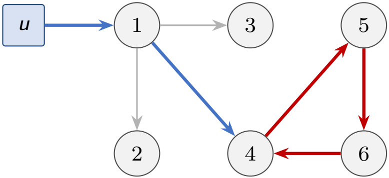
- Four non-zero entries in A^2, no others. Nodes 2 and 3 have no edge out $\to$ no walk of any length continues past them
- No diagonal entry in A^2 $\to$ no closed walk of length 2. The one cycle here needs three steps

One walk out of the driver, three inside the cycle. **Which of the four survives at $k = 3$?**

A^3: The Cycle Lands on the Diagonal

- **A^3, four non-zero entries**

$$[A^3]_{44} = [A^3]_{55} = [A^3]_{66} = a_{46}a_{65}a_{54}, \qquad [A^3]_{61} = a_{65}a_{54}a_{41}$$



One cycle, read from each of its three nodes:

$$4 \to 5 \to 6 \to 4$$

$$5 \to 6 \to 4 \to 5$$

$$6 \to 4 \to 5 \to 6$$

the same three edges, the same product $\to$ three equal entries

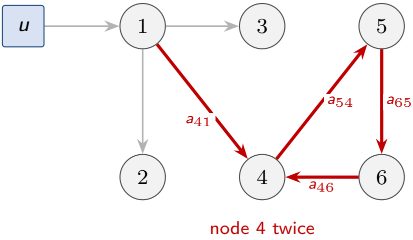
- $[A^k]_{ii} \neq 0 \iff$ a **closed walk** of length k through node i. The shortest such walk is a **cycle**
- Nodes 1, 2, 3 carry no diagonal entry at any k $\to$ no cycle passes through them
- $[A^3]_{61}$ $\to$ the walk $1 \to 4 \to 5 \to 6$, three steps out of the driver

Which nodes of a graph can ever appear on the diagonal of A^k?

A^4: The Same Ends, Two Lengths

▶ **The same row and the same column, two lengths**

$[A]_{41} = a_{41}$ $(1 \to 4,$ length $1)$ $[A^4]_{41} = a_{46}a_{65}a_{54}a_{41}$ $(1 \to 4 \to 5 \to 6 \to 4,$ length $4)$



Cycle weight $P = a_{46}a_{65}a_{54} \;\to\; [A^4]_{41} = P\,a_{41}$

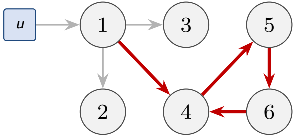
▶ A **walk**, not a path: node 4 is visited at step 1 and again at step 4

Same start, same end, one extra lap. **Where does the extra factor P come from?**

One Lap Later: Every Entry Times the Cycle Weight

- **A^4, four non-zero entries, one lap added to each**

$$[A^4]_{41} = P\,a_{41}, \quad [A^4]_{46} = P\,a_{46}, \quad [A^4]_{54} = P\,a_{54}, \quad [A^4]_{65} = P\,a_{65}$$



the four edges A^4 lands on

One lap around the cycle $\rightarrow$ multiply by P and land back on the length-1 positions that have an edge out, which nodes 2 and 3 do not

- Non-zero again at $k = 7, 10, \ldots$, one more lap each time: $[A^7]_{41} = P^2 a_{41}$
- $A^k \neq 0$ for every k $\rightarrow$ the walk list never ends, because the cycle can always be run again
- No cycle anywhere $\rightarrow$ every walk is a path $\rightarrow$ walks stop at length $n-1$ $\rightarrow$ $A^n = 0$

The **controllability matrix** is exactly this list of walks, column by column. Next section.

Chapter 2 Contents

Rank: A Spanning Question

CONCEPT Rank

- $\operatorname{rank} M$ = number of **independent columns** = $\dim$(column space)
- $Mz = w$ solvable $\iff$ $w \in \operatorname{im} M$, uniquely $\iff$ also $\ker M = \{0\}$
- $\operatorname{rank} M + \dim \ker M$ = number of columns

EXAMPLE A worked 3×3

$$M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \qquad \text{Row 3} = \text{Row 1} + \text{Row 2} \implies \operatorname{rank} M = 2$$

- Three columns spanning a **plane**, not $\mathbb{R}^3$ $\rightarrow$ column 3 $= 2$(column 1) + column 2, one column redundant
- $\ker M$ spanned by $[-2 \quad -1 \quad 1]^\top$, one direction lost $\rightarrow$ check: $M[-2 \quad -1 \quad 1]^\top = 0$

Rank never grows under multiplication: $\operatorname{rank}(MN) \leq \min\{\operatorname{rank} M, \operatorname{rank} N\}$. **Do these columns span $\mathbb{R}^n$?** $\rightarrow$ the one question controllability asks (Ch. 1).

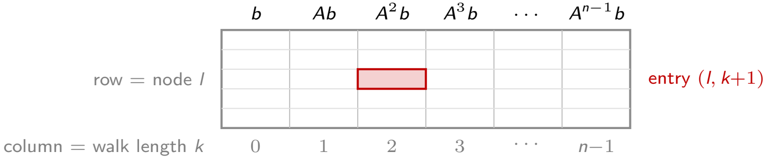
The Controllability Matrix: One Column per Walk Length

$$\mathcal{C} = \begin{bmatrix} b & Ab & A^2 b & \cdots & A^{n-1} b \end{bmatrix}, \qquad \text{controllable} \iff \operatorname{rank} \mathcal{C} = n$$

Kalman rank condition (Ch. 1) · n = number of states · one column per walk length

▶ The mechanism

- Single input, driver node 1 $\to$ $b = [1\ 0\ 0\ 0\ 0\ 0]^\top$ $\to$ $A^k b =$ **column 1 of A^k**
- Entry $(l, k{+}1)$ of $\mathcal{C}$ = the sum of the weight products of all k-step walks from the **driver** node to node l
- The input edge carrying weight 1 $\to$ the count starting at the driver, not at u

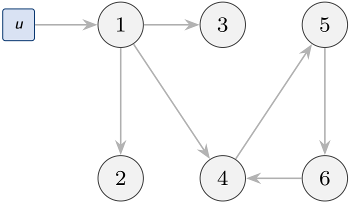


One column = one walk length. **How far out of the driver can the columns get, and what decides that, the numbers or the picture?**

Columns 1 and 2: b and Ab

Column 1 $= b \rightarrow$ length 0

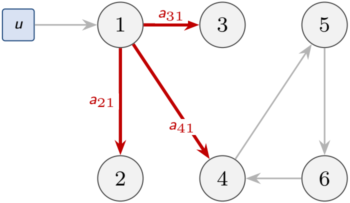
$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



node 1, the driver. No step taken yet

Column 2 $= Ab \rightarrow$ length 1

$$Ab = \begin{bmatrix} 0 \\ a_{21} \\ a_{31} \\ a_{41} \\ 0 \\ 0 \end{bmatrix}$$



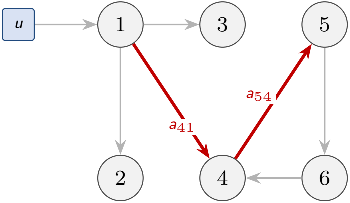
nodes 2, 3 and 4, one step out of the driver

Column $k+1$ listing the nodes reached in exactly k steps. **Which nodes are still dark after one step?**

Columns 3 and 4: A^2b and A^3b

Column 3 $= A^2b \;\rightarrow\;$ length 2

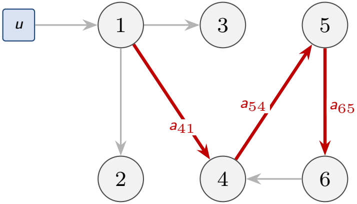
$$A^2b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_{54}a_{41} \\ 0 \end{bmatrix}$$



node 5, by the walk $1 \to 4 \to 5$

Column 4 $= A^3b \;\rightarrow\;$ length 3

$$A^3b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_{65}a_{54}a_{41} \end{bmatrix}$$



node 6, the last node reached for the first time

Each column reaching one step further from the driver. At length 3 the stem is exhausted. **What can columns 5 and 6 still add?**

Six Nodes, Six Columns: $\mathcal{C}$ in Full

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{21} & 0 & 0 & 0 & 0 \\ 0 & a_{31} & 0 & 0 & 0 & 0 \\ 0 & a_{41} & 0 & 0 & P\,a_{41} & 0 \\ 0 & 0 & a_{54}a_{41} & 0 & 0 & P\,a_{54}a_{41} \\ 0 & 0 & 0 & a_{65}a_{54}a_{41} & 0 & 0 \end{bmatrix}$$

columns $= b,\ Ab,\ A^2b,\ A^3b,\ A^4b,\ A^5b$ · rows $=$ nodes 1 to 6 · cycle weight $P = a_{46}a_{65}a_{54}$

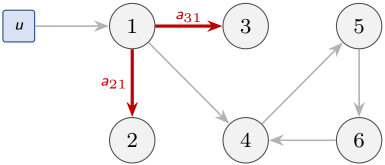
- Columns 1 to 4 covering every node once: 1, then $2, 3, 4$, then 5, then 6
- Column 5 returning to node 4, once around the cycle: $P\,a_{41} = a_{46}a_{65}a_{54}a_{41}$ $\rightarrow$ a walk, not a path
- Column 6 returning to node 5, equal to $P \times$ (column 3) $\rightarrow$ nothing new
- Every non-zero entry a **single product** of weights, one walk each. No entry a sum $\rightarrow$ no two walks of equal length sharing the same ends on this graph

A row is a node, a column is a walk length out of the driver. The matrix is the picture, written out.

Rank From the Picture: Two Nodes, One Ratio

- **Where does the rank stop, and why?**
  - Row 2 $= [0 \;\; a_{21} \;\; 0 \;\; 0 \;\; 0 \;\; 0]$ and row 3 $= [0 \;\; a_{31} \;\; 0 \;\; 0 \;\; 0 \;\; 0]$, **proportional** for every realization $\Rightarrow \; \operatorname{rank}\mathcal{C} \leq 5 < 6$
  - Column 2 the only column with a non-zero entry in rows 2 and 3 $\rightarrow$ one column carrying the whole story of nodes 2 and 3

Graph reason: nodes 2 and 3 at the same walk length from the driver, each reached by exactly one walk, neither with an edge out $\rightarrow$ the input reaching them only together, in the fixed ratio $a_{21} : a_{31}$



ratio $a_{21} : a_{31}$ fixed

Same walk length, one walk each, no edge out $\rightarrow$ the input never telling node 2 from node 3, whatever the weights are. **So how large can the rank actually be?**

Rank Exactly Five: The Determinant, Computed

- **Rows $1, 2, 4, 5, 6$ of $\mathcal{C}$ by columns 1 to 5**

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_{21} & 0 & 0 & 0 \\ 0 & a_{41} & 0 & 0 & P\,a_{41} \\ 0 & 0 & a_{54}\,a_{41} & 0 & 0 \\ 0 & 0 & 0 & a_{65}\,a_{54}\,a_{41} & 0 \end{bmatrix}$$

$$\det \;=\; a_{21}\,a_{41}^{3}\,a_{54}^{3}\,a_{65}^{2}\,a_{46} \;\neq\; 0 \qquad \Longrightarrow \qquad \operatorname{rank}\mathcal{C} = 5$$

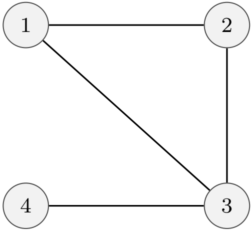
- A single product of edge weights, every factor non-zero $\rightarrow$ $\operatorname{rank}\mathcal{C} = 5$ for **every** realization, not for almost all
- Column 5 isolating node 4 and lifting the rank from 4 to 5. Column 6 a multiple of column 3, adding nothing
- No weight chosen, no number measured, only which entries are non-zero $\rightarrow$ the rank fixed by **edge existence**, not by **edge weight**

$\operatorname{rank}\mathcal{C} = 5 < 6$ for every realization, read off the picture alone. **Part II** gives the missing direction a name.

Chapter 2 Contents

Four Agents on a Network: The Setup

EXAMPLE **Four agents, unit weights**



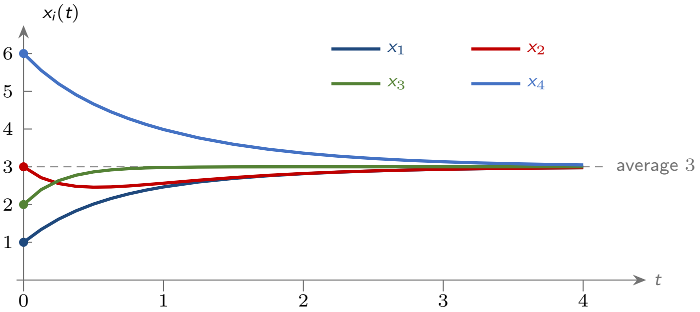
$$x_{\text{init}} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 6 \end{bmatrix}$$

entry i $\to$ agent i's starting number

- **Agent** (one robot each, holding one number) $\to$ one node, one state x_i
- Undirected edges 12, 23, 34, 13, every weight 1 $\to$ $w_{ij} = w_{ji} = 1$
- Neighbor counts $2, 2, 3, 1$ $\to$ node 3 sees three others, node 4 only one
- No input, no leader $\to$ nothing outside the network sets a value

[Question] Four different numbers, nobody in charge. **What could they possibly agree on?**

Four Agents, One Value: The Trajectories



- All four curves **meet at one value**
- That value is $3 = (1 + 3 + 2 + 6)/4$, the **average** of the starting numbers
- Agent 2 starts *at* the average and still dips to 2.459 at $t = 0.522$

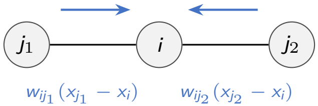
[Question] Four different starting numbers, one common limit.
Why that value, and how fast?

The Rule Each Agent Follows: $\dot{x} = -\mathcal{L}x$

$$\dot{x}_i = \sum_{j \in N(i)} w_{ij}\,(x_j - x_i) \qquad \Longleftrightarrow \qquad \dot{x} = -\mathcal{L}x$$

CONCEPT Laplacian $\mathcal{L} = D - W$

- $N(i)$ $\rightarrow$ the **neighbors** of node i, the nodes joined to it by an edge
- W $\rightarrow$ the **symmetric adjacency matrix**, $w_{ij} = w_{ji} > 0$ on an edge and 0 otherwise
- $D = \mathrm{diag}(W\mathbb{1})$ $\rightarrow$ the **degree matrix** (unrelated to the feedthrough D of the LTI model) $\rightarrow$ $\mathcal{L} = D - W$

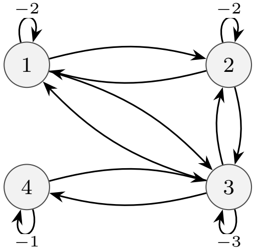


Each neighbor pulls node i toward its own value. The pulls add $\rightarrow$ $\dot{x}_i$

Every term is a **difference**. No agent knows the average $\rightarrow$ each one sees only its neighbors.

The Laplacian of This Graph: Every Row Sums to Zero

EXAMPLE Four agents, Sec. 2.1 definition applied



the graph of $\dot{x} = -\mathcal{L}x$

$$\mathcal{L} = D - W = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Each undirected edge $\to$ two opposite directed edges of weight 1. Each node $\to$ a **self-loop** of $-$degree. Degrees $2, 2, 3, 1$. No input $\to$ $-\mathcal{L}$ in the role of A

- Row 3 against the local rule: $N(3) = \{1, 2, 4\}$ $\to$ $\dot{x}_3 = (x_1 - x_3) + (x_2 - x_3) + (x_4 - x_3) = x_1 + x_2 - 3x_3 + x_4$, which is row 3 of $-\mathcal{L}$
- Row 4: $N(4) = \{3\}$ $\to$ $\dot{x}_4 = x_3 - x_4$, degree 1 $\to$ the one node with a single neighbor

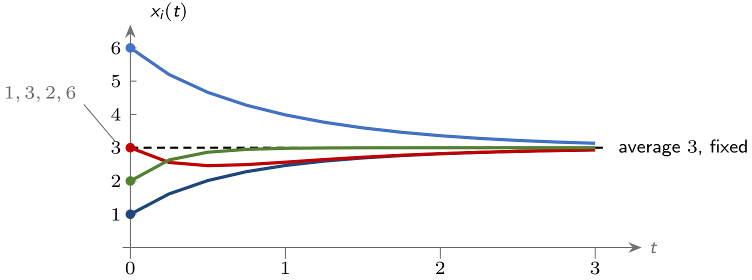
Every row sums to zero $\to$ $\mathcal{L}\mathbf{1} = 0$. **Agreement is an equilibrium** before any computation.

Reading the Curves: The Average Never Moves

- **One common limit** $\rightarrow$ agreement. A single number, not a schedule

$$\frac{d}{dt}(\mathbf{1}^\top x) = \mathbf{1}^\top \dot{x} = -\mathbf{1}^\top \mathcal{L} x = 0, \qquad \mathbf{1}^\top \mathcal{L} = (\mathcal{L}\mathbf{1})^\top = 0 \text{ since } \mathcal{L} = \mathcal{L}^\top$$

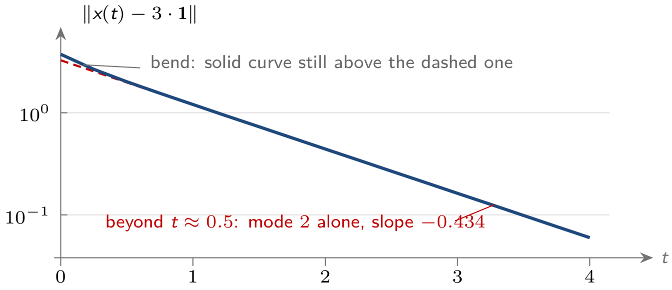
- The sum $\mathbf{1}^\top x$ is constant $\rightarrow$ the limit is **forced** to be the initial average, here $(1+3+2+6)/4 = 3$



$\mathbf{1}^\top x$ constant $\rightarrow$ the agreement **value** is decided at $t = 0$, not by the dynamics.

Reading the Curves: Only the Spread Is Monotone

- Individual states are **not monotone**: agent 2 starts at the average and still dips to 2.459 at $t = 0.522$
- Only the **spread** $\|x(t) - 3 \cdot \mathbf{1}\|$ falls at every instant $\rightarrow$ slope -0.539 over $[0, 0.5]$, then straight at -0.434



[Question] The average never moves. Why should the **spread** go to zero at all, and why does the line **bend** before it straightens?

Why It Happens: One Zero Mode, the Rest Decay

CONCEPT Matrix Exponential

- $\dot{x} = Ax$, initial value $x(0)$ ($= x_{\text{init}}$, the initial state of Ch. 1) $\rightarrow$ $x(t) = e^{At}x(0)$, where $e^{At} := \sum_{j\geq 0}(At)^j/j!$
- Eigenpair $Av = \lambda v$, $v \neq 0$ $\rightarrow$ $e^{At}v = e^{\lambda t}v$, term by term in the series
- Here $A = -\mathcal{L}$: $\mathcal{L}v = \lambda v$ $\rightarrow$ $e^{-\mathcal{L}t}v = e^{-\lambda t}v$

- $\mathcal{L} = \mathcal{L}^\top$ $\rightarrow$ real eigenvalues, orthogonal eigenvector basis $\rightarrow$ $x(t) = \sum_k c_k\, e^{-\lambda_k t} v_k$, $\quad c_k = \langle x(0), v_k\rangle / \langle v_k, v_k\rangle$

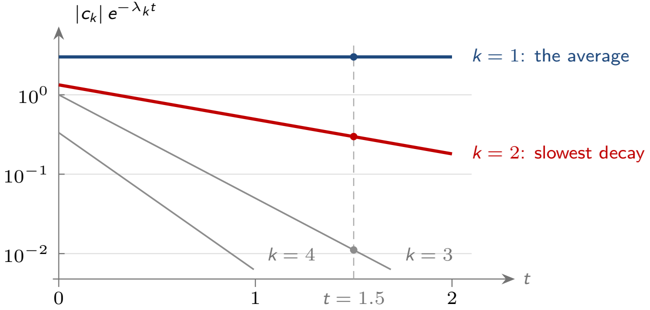
k	λ_k	v_k	c_k	
1	0	$(1,1,1,1)$	$12/4 = 3$	$\rightarrow$ never decays $=$ the average
2	1	$(1,1,0,-2)$	$-8/6 = -4/3$	$\rightarrow$ slowest, sets the tail
3	3	$(1,-1,0,0)$	$-2/2 = -1$	
4	4	$(1,1,-3,1)$	$4/12 = 1/3$	

- Trace check: $0+1+3+4 = 8 = 2+2+3+1$, the degrees

[Question] Which of the four terms is still visible at $t = 2$?

Rate: The Slowest Surviving Mode

- At $t = 1.5$ the four mode sizes are $3,\ 0.298,\ 0.011,\ 0.0008$ $\rightarrow$ mode 2 is about 27 times mode 3
- Disagreement decays at rate $\lambda_2 = 1$, the **second-smallest** eigenvalue of $\mathcal{L}$ (its algebraic connectivity), and $-\lambda_2/\ln 10 = -0.434$ is exactly the tail slope read off the log plot

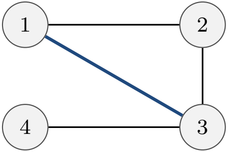


Zero eigenvalue $\rightarrow$ **the value** they agree on.
Second-smallest eigenvalue $\rightarrow$ **the time** it takes.

Connected: What $\lambda_2 > 0$ Means

CONCEPT Connected

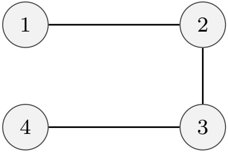
- **Connected** $\rightarrow$ an undirected path between every pair of nodes (path = a walk visiting no node twice, Sec. 2.2)
- connected $\iff \operatorname{rank}\mathcal{L} = n - 1 \iff \lambda_2 > 0$
- $\lambda_2 = 0$ $\rightarrow$ the graph splits $\rightarrow$ one agreement value per piece, never a common one



with the chord $1{-}3$

$\{0, 1, 3, 4\}$, $\operatorname{rank}\mathcal{L} = 3$

$\rightarrow\ \lambda_2 = 1$



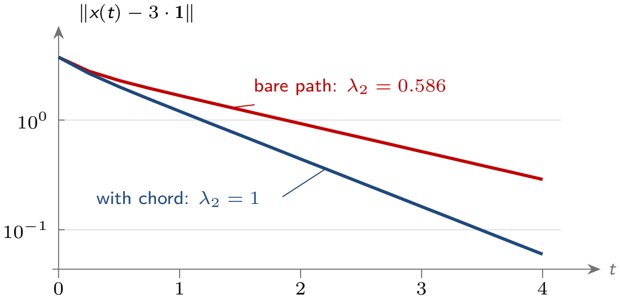
chord deleted, bare path

$\{0,\ 2{-}\sqrt{2},\ 2,\ 2{+}\sqrt{2}\}$, $\operatorname{rank}\mathcal{L} = 3$

$\rightarrow\ \lambda_2 = 0.586$

Both graphs are connected $\rightarrow$ one zero eigenvalue each, one agreement value each. The **second** eigenvalue is where they differ.

One Edge Added: Faster Agreement



- With the chord: $\lambda_2 = 1 \;\rightarrow\;$ log-slope -0.434 per unit t
- Bare path: $\lambda_2 = 0.586 \;\rightarrow\;$ log-slope -0.254
- $1/0.586 = 1.71 \;\rightarrow\;$ the same four agents take about 1.7 times as long

One edge added $\rightarrow$ the line **tilts**. Algebraic connectivity is a **rate**, not only a yes or no.

Summary and Next Week: Ch. 3 Controllability

▶ Chapter 2 in one breath

state model → its **graph** → walks and powers → controllability matrix → consensus

- A linear system **is** a weighted directed graph. The entries of A^k and of $\mathcal{C}$ are **walks** on it

▶ Flag for Part II: dependent entries

- Entries of a Laplacian are **dependent**: each diagonal entry is the negative sum of the **off-diagonal** entries in its row, so every row sums to zero. A generic A is free, a Laplacian is not
- **Part II** treats entries as **free parameters** $\rightarrow$ Laplacian patterns must respect that dependency

[Next week] Ch. 3, the **Kalman rank condition**: the first yes-or-no answer, every number still known.
"Can the input **steer every state**, and can the output **reveal** it?"