Linear Systems on Graphs

Controllability of Structured Networks

Paper Talks

How to Present a Paper: One Example Talk

Example talk: Controllability of Complex Networks Liu et al., Nature 2011

Nam-Jin Park

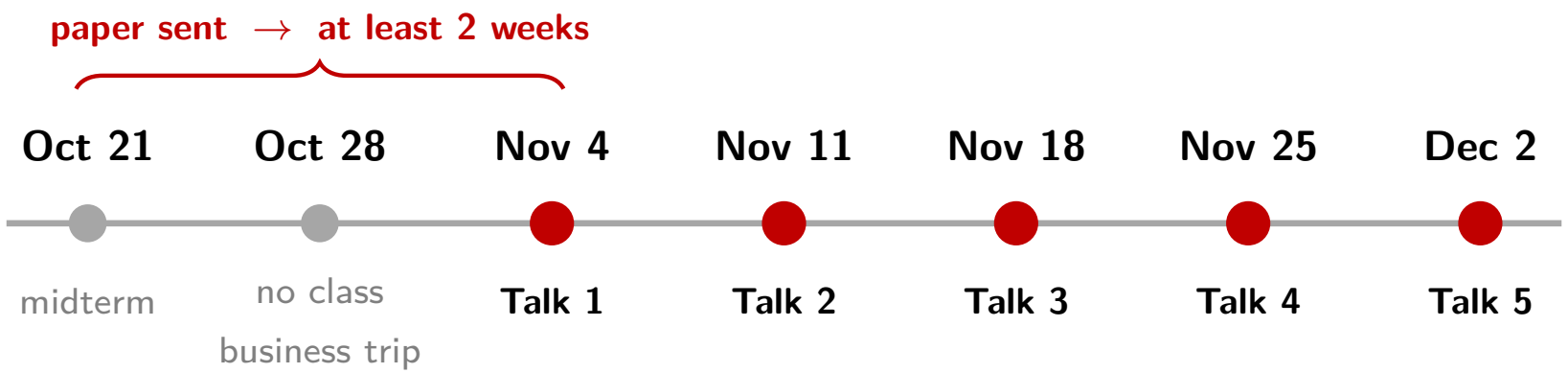
PUKYONG NATIONAL UNIVERSITY

Paper Talks: 5 Teams, 5 Weeks

CONCEPT Teams

- Tien, Jubong $\rightarrow$ **individual**
- Alex, Claus, Peter, Emily, Kine, Tu $\rightarrow$ **3 teams of 2**

- **Schedule: 1 team per week, at least 2 weeks to prepare**



Choosing a Paper: Your Choice, Then My Check

CONCEPT Requirements

- **Regular paper** in a **reputable journal**, **at least 6 pages**
- Talk length $\rightarrow$ **at least 20 minutes**

How to Choose

- Any topic you care about $\rightarrow$ fine, if it meets the requirements
- Today's example paper $\rightarrow$ also fine, in more depth
- **By Oct 21** $\rightarrow$ send me your team, its paper, and its preferred week
- Then $\rightarrow$ my check $\rightarrow$ prepare

Today's example talk shows the format. **Which paper will your team send by Oct 21?**

Talk Outline: Parts 0 to 6

CONCEPT The Rule

- **Every student must follow your talk** $\rightarrow$ intuition and examples

- **0 Paper Title** $\rightarrow$ title, authors, journal, year
- **1 Motivation** $\rightarrow$ why the main result is needed, and what it solves
- **2 Preliminaries** $\rightarrow$ the concepts it needs, each with a small example
- **3 Main Result** $\rightarrow$ the statement, then its intuitive meaning
- **4 Example** $\rightarrow$ a graph the other students can work through by hand
- **5 Application** $\rightarrow$ where the result can be used
- **6 Conclusion** $\rightarrow$ limitations, and an idea to improve them

Papers sell their strengths. **Which weakness will your team find?**

0 Paper Title: Minimum Inputs by Maximum Matching

Controllability of Complex Networks

Yang-Yu Liu, Jean-Jacques Slotine, and Albert-László Barabási

Nature 473, 167–173, 2011 **EXAMPLE TALK**

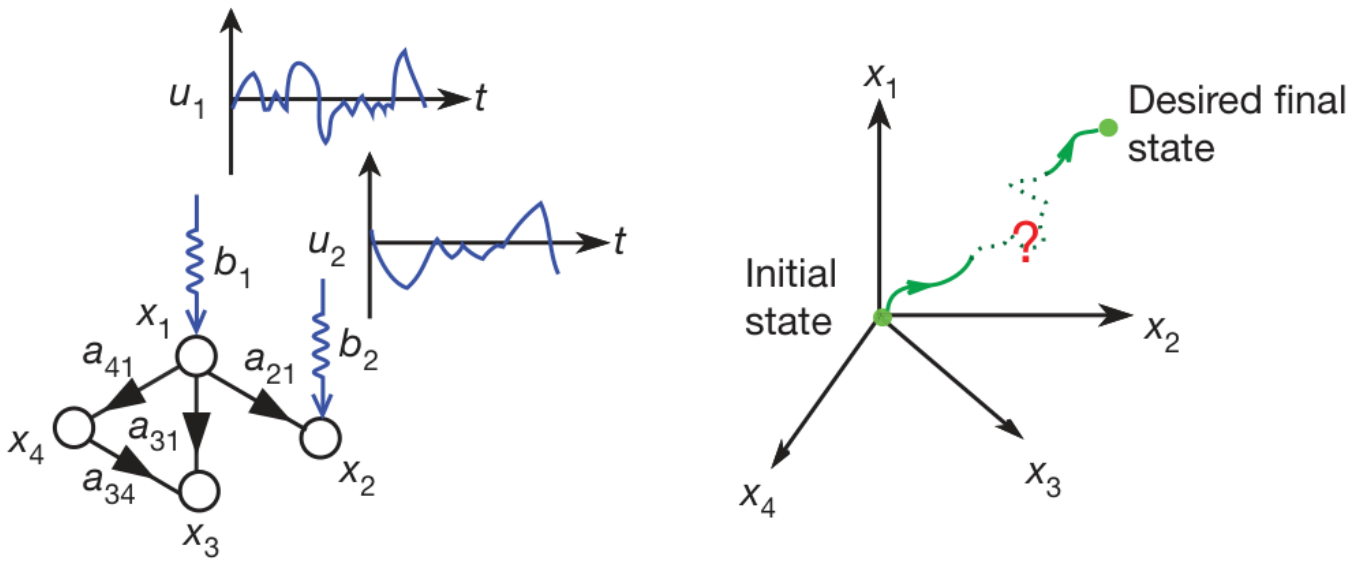


Fig. 1a (p. 168): 2 input signals $u_1(t)$, $u_2(t)$ at nodes x_1, x_2 steer 4 nodes from an initial to a desired final state

SI = the paper's Supplementary Information (44 pages): definitions, theorems, proofs

Here 2 inputs control 4 nodes. **Can Lin's theorem tell us that 2 is the minimum?**

1 Motivation: How Many Inputs, and Where?

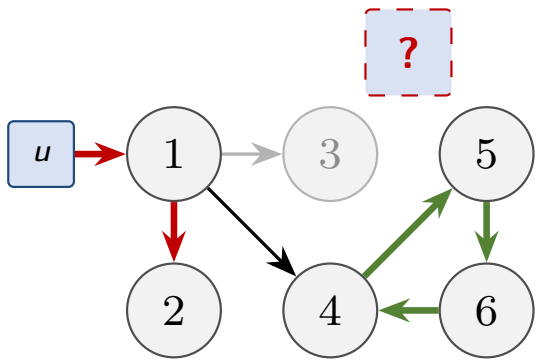
THEOREM Lin (Ch. 2)

A graph is **SC** if and only if

(C1) **input-connected**

(C2) a **disjoint set of stems and cycles** covers all state nodes

Hosoe (Ch. 2): $\operatorname{rank} \mathcal{C}$ = the **maximum** number of **reachable** state nodes covered by (C2)



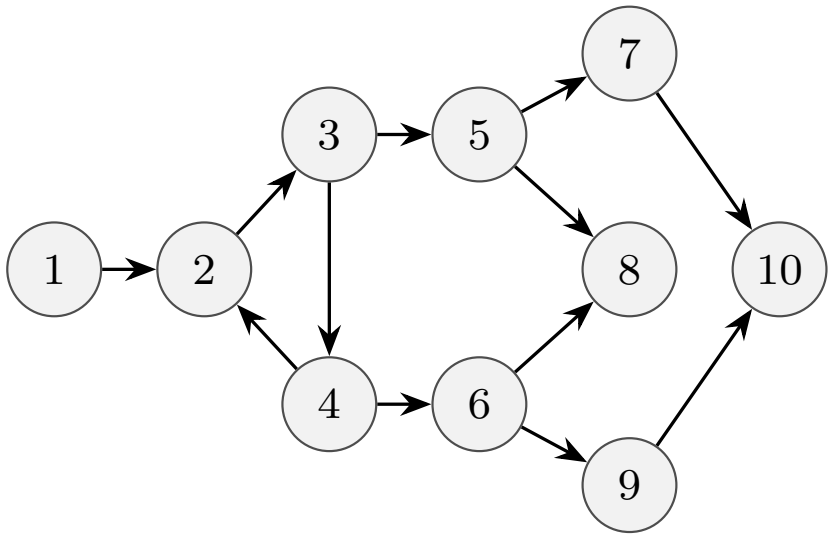
stem + **cycle** $\rightarrow$ $\operatorname{rank} \mathcal{C} = 5 < 6$

- Lin, Hosoe $\rightarrow$ check **given** inputs
- Design $\rightarrow$ **how many** inputs, **where**?
- Input = actuator = cost $\rightarrow$ **minimum**
- By eye: node 3 left out $\rightarrow$ **minimum 2**

On 6 nodes the minimum is easy to see. **Can you see it on 10 nodes?**

1 Motivation: The Minimum on 10 Nodes?

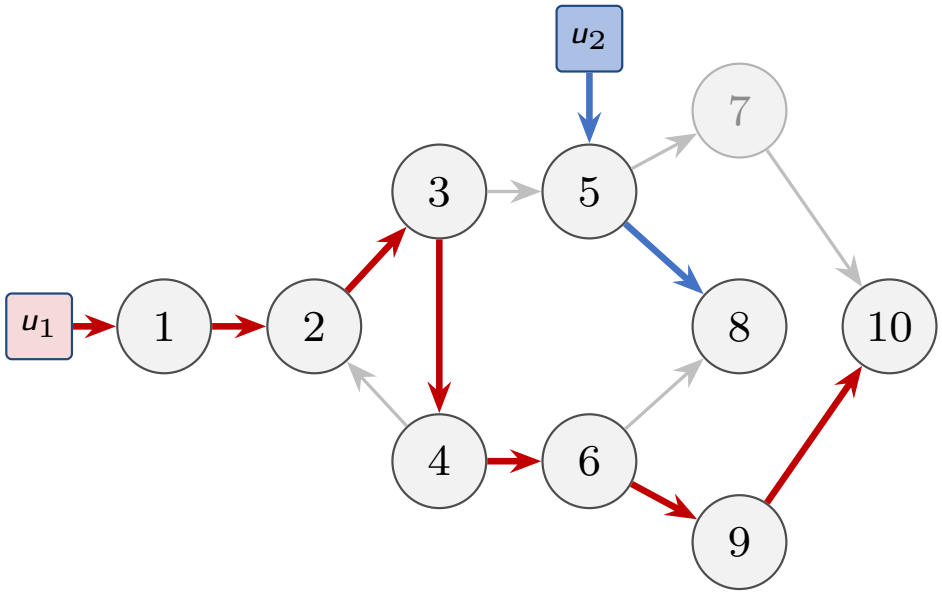
- **Find the minimum number of inputs for SC, and where they go**



10 nodes, 12 edges, no input yet. **How many inputs are needed, and where?**

1 Motivation: The Minimum on 10 Nodes?

▶ **Find the minimum number of inputs for SC, and where they go**



Guess: 2 inputs, at 1 and 5

Stem from u_1: 1, 2, 3, 4, 6, 9, 10

Stem from u_2: 5, 8

9 nodes covered, **7 left out**

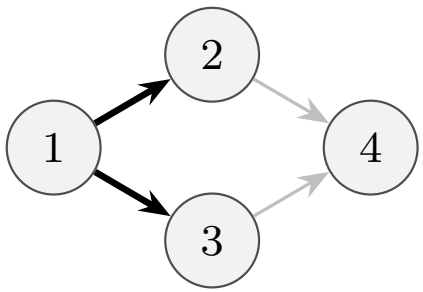
$\rightarrow \operatorname{rank} \mathcal{C} = 9 < 10$ (Hosoe)

Guessing fails on 10 nodes. **Is there a simple way to find the minimum?**

2 Preliminaries: Matching on a 4-Node Graph

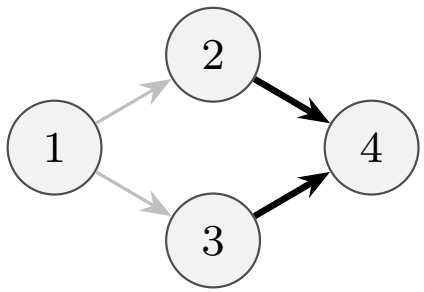
CONCEPT Matching (SI Definition 8, p. 14)

- **MATCHING** → a set of edges **sharing no start node and no end node**
- **MATCHED** → end of a matching edge · **UNMATCHED** → the rest



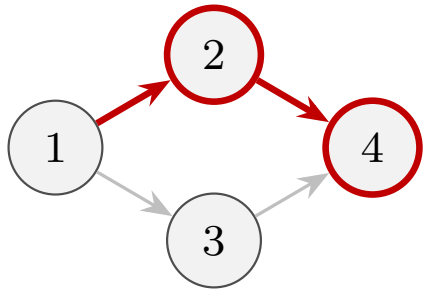
Not a matching

$1 \to 2$, $1 \to 3$: same start node



Not a matching

$2 \to 4$, $3 \to 4$: same end node



A matching (red)

matched 2, 4 · unmatched 1, 3

Matchings with 2 edges: $\{1 \to 2,\ 2 \to 4\}$, $\{1 \to 3,\ 3 \to 4\}$, $\{1 \to 2,\ 3 \to 4\}$, $\{1 \to 3,\ 2 \to 4\}$

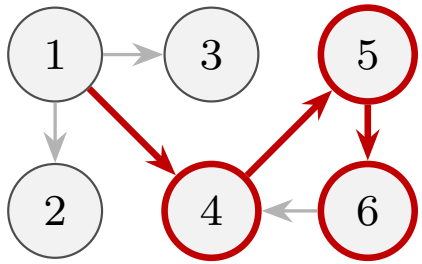
Here no matching has more than 2 edges. **When is a matching as large as it can be?**

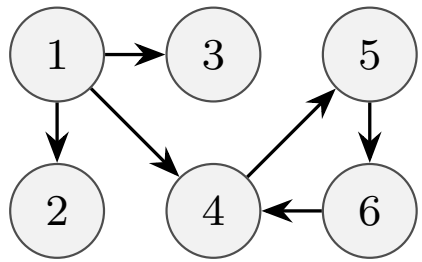
2 Preliminaries: When Is a Matching Maximum?

CONCEPT Maximum Matching (SI Definition 8, p. 14)

- **MAXIMUM MATCHING** $\rightarrow$ a matching with the most edges

A





3 edges, and no edge can be added

red edge: matching edge · red border: matched node

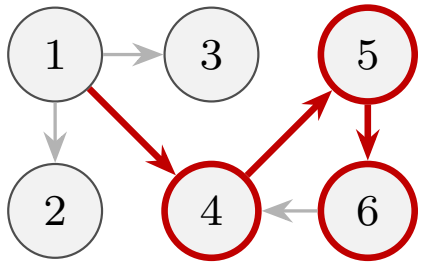
No edge can be added to matching A. **Is A then a maximum matching?**

2 Preliminaries: When Is a Matching Maximum?

CONCEPT Maximum Matching (SI Definition 8, p. 14)

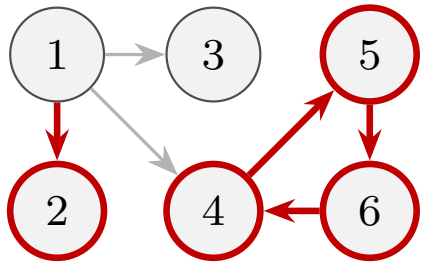
- **MAXIMUM MATCHING** → a matching with the most edges

A



3 edges, and no edge can be added

B



Maximum: only 1, 4, 5, 6 have an edge out

red edge: matching edge · red border: matched node

Matching B has 4 edges, A only 3. **How many inputs does the network need?**

3 Main Result: Minimum Driver Nodes = Unmatched Nodes

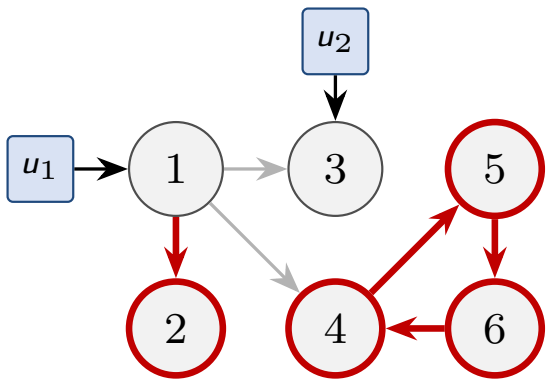
THEOREM Minimum Input Theorem (SI Theorem 2, p. 14–15)

DRIVER NODE $\rightarrow$ a node with its own input signal

N nodes, $|M^*|$ = size of a maximum matching

$$N_D = \max\big(N - |M^*|,\ 1\big) \quad = \text{minimum number of driver nodes}$$

Driver nodes $\rightarrow$ the **unmatched** nodes $\cdot$ all matched (**perfect**) $\rightarrow$ $N_D = 1$



$N = 6, \quad |M^*| = 4 \Rightarrow N_D = 2$

driver nodes $\{1, 3\}$, or $\{1, 2\}$

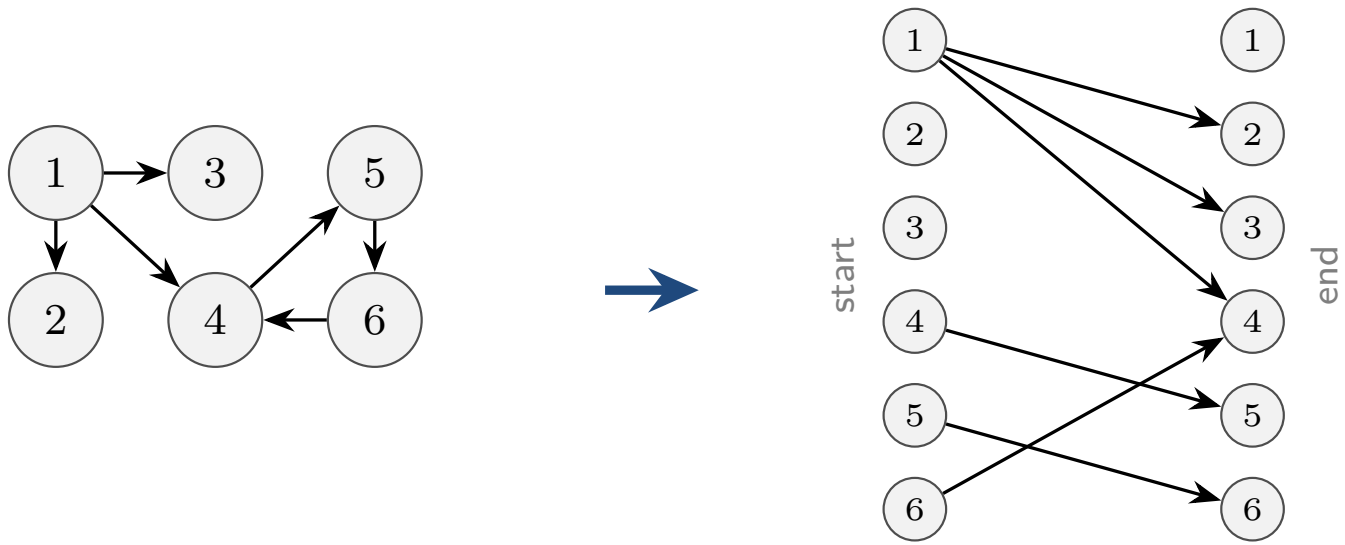
Ch. 2 check: inputs at 1, 3 $\rightarrow$ $\operatorname{rank} \mathcal{C} = 6$, SC

A cycle of matching edges that no stem reaches shares an input signal with a driver node (SI p. 15)

The six-node network needs 2 inputs. **Why must they enter unmatched nodes?**

3 Main Result: Matching Edges Become Stems and Cycles

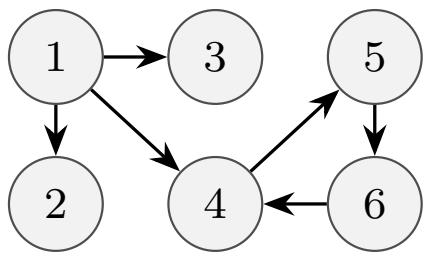
- **BIPARTITE GRAPH** $\rightarrow$ 2 node sets, edges only between them
- Here $\rightarrow$ each node twice, start (left), end (right) $\cdot$ $j \rightarrow i \Rightarrow$ start j to end i
- **Match** $\rightarrow$ no shared start, no shared end $\rightarrow$ keep **as many edges as possible**
- **Back on the graph** $\rightarrow$ the matching edges join up $\rightarrow$ **stems**, **cycles**



Each edge now runs from a start node to an end node. **How many edges can we keep?**

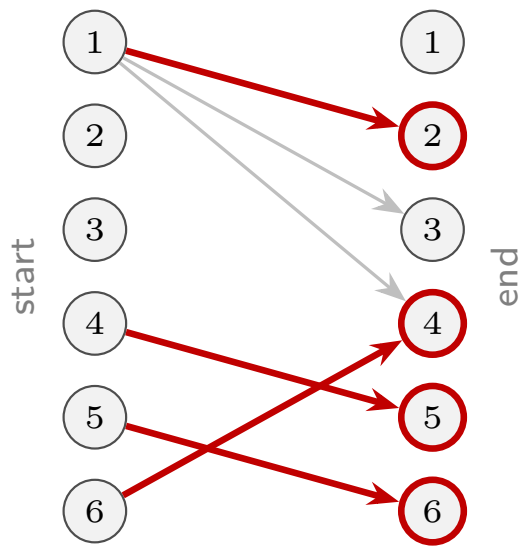
3 Main Result: Matching Edges Become Stems and Cycles

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- **Match** $\rightarrow$ no shared start, no shared end $\rightarrow$ keep **as many edges as possible**
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4 matching edges, ends 2, 4, 5, 6

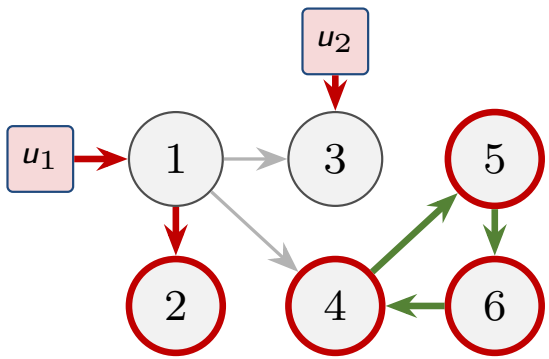
unmatched 1, 3 $\rightarrow$ own inputs

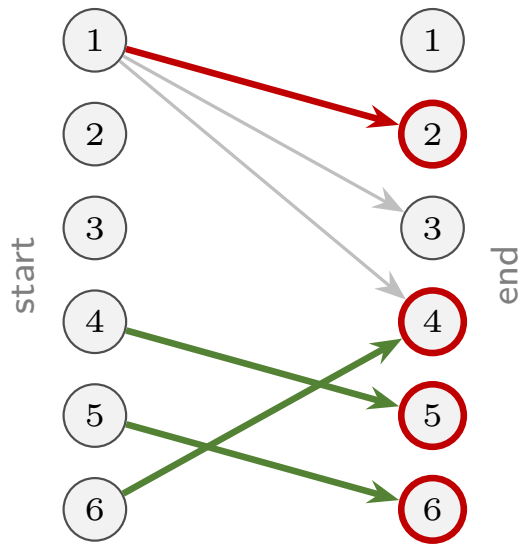


4 edges kept, nodes 1 and 3 left unmatched. **What do the 4 edges form on the graph?**

3 Main Result: Matching Edges Become Stems and Cycles

- **BIPARTITE GRAPH** $\rightarrow$ 2 node sets, edges only between them
- Here $\rightarrow$ each node twice, start (left), end (right) $\cdot$ $j \rightarrow i$ $\Rightarrow$ start j to end i
- **Match** $\rightarrow$ no shared start, no shared end $\rightarrow$ keep **as many edges as possible**
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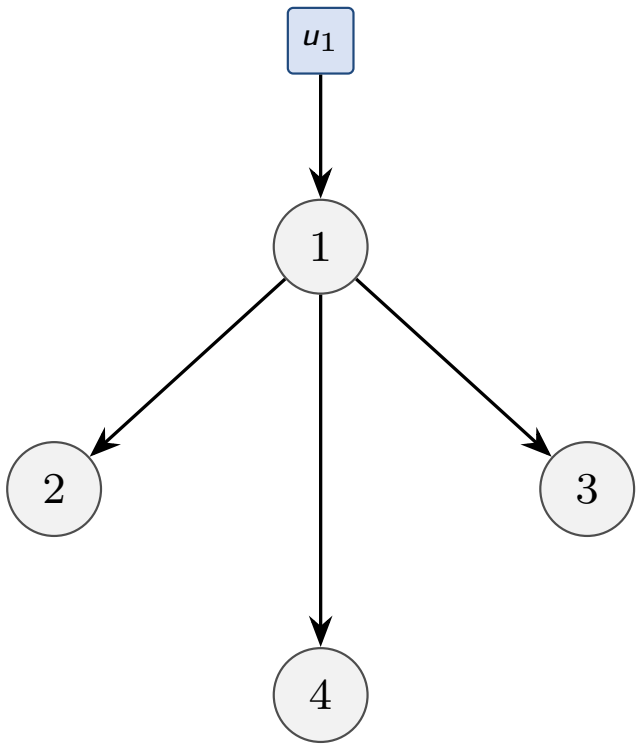
2 stems + 1 cycle $\rightarrow$ Lin (C2)

edge $1 \rightarrow 4$ reaches the cycle $\rightarrow$ SC

Each unmatched node starts its own stem. **How many inputs does a star need?**

4 Example: The Star, How Many Inputs and Where?

- **Directed star: node 1 → nodes 2, 3, 4 (Fig. 1e, p. 168)**

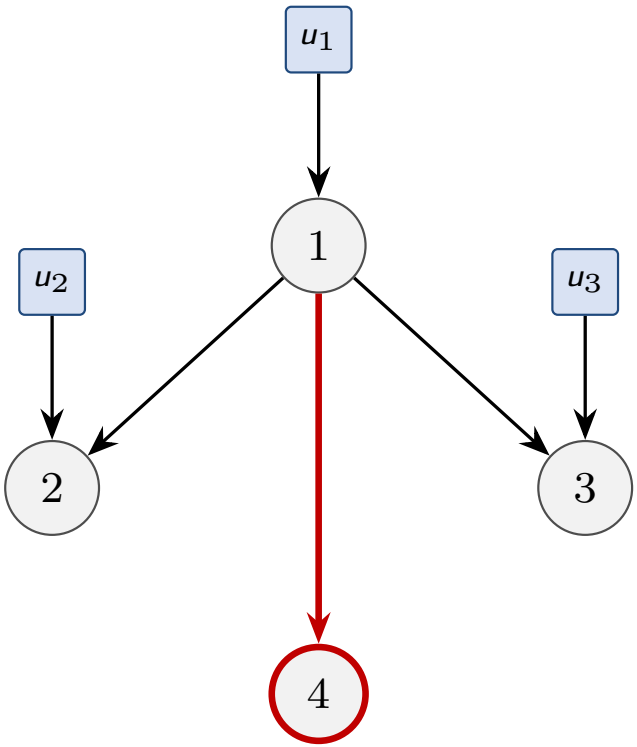


Recall: $N_D = \max\big(N - |M^*|, 1\big)$

Input u_1 reaches every node. **How many inputs does the star need, and where?**

4 Example: The Star, How Many Inputs and Where?

▶ **Directed star: node 1 → nodes 2, 3, 4 (Fig. 1e, p. 168)**



Recall: $N_D = \max(N - |M^*|, 1)$

Node 1: at most 1 matching edge out

$\rightarrow \ |M^*| = 1 \ \rightarrow \ N_D = 4 - 1 = 3$

node 1 + any 2 of nodes 2, 3, 4 $\rightarrow$ **3 driver sets**

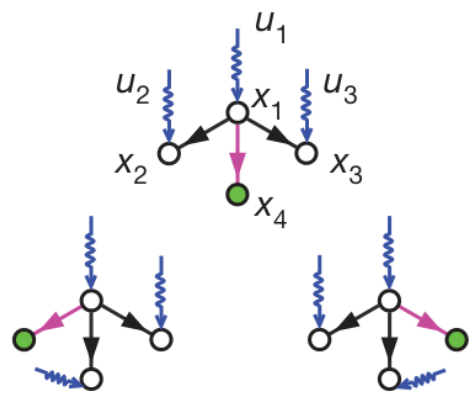


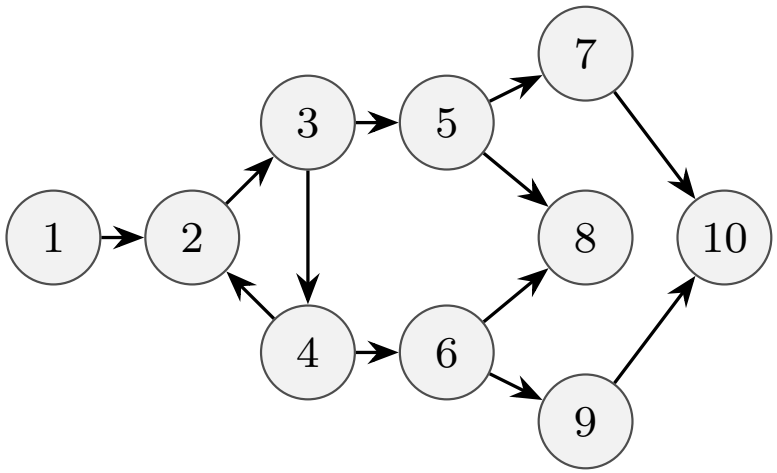
Fig. 1f (p. 168): the 3 maximum matchings

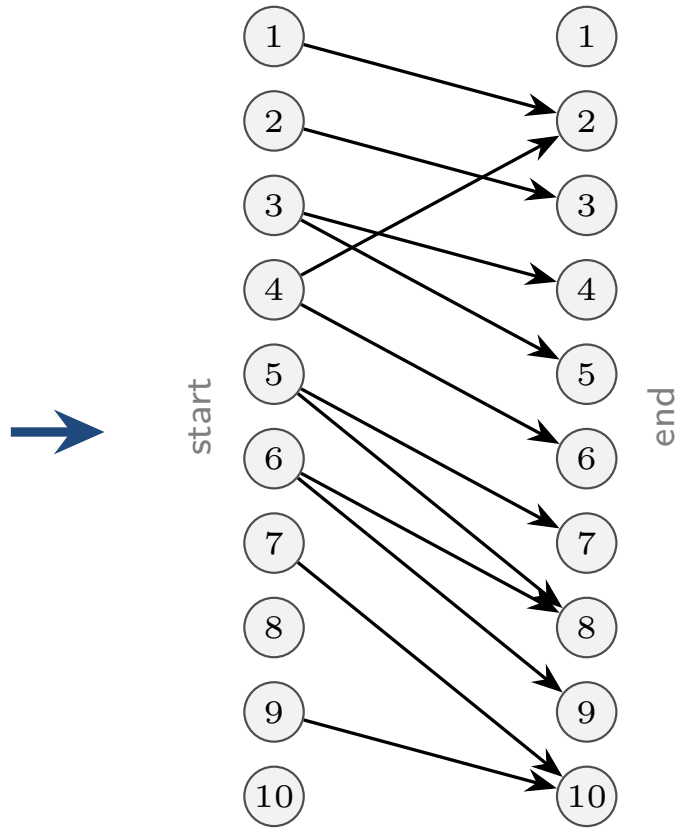
paper: purple = matching edge, green = matched

The star needs 3 inputs, placed in 3 ways. **Can we now solve the 10-node graph?**

4 Example: The 10-Node Graph, Solved by Matching

▶ **Step 1: a maximum matching → Step 2: inputs at the unmatched nodes**

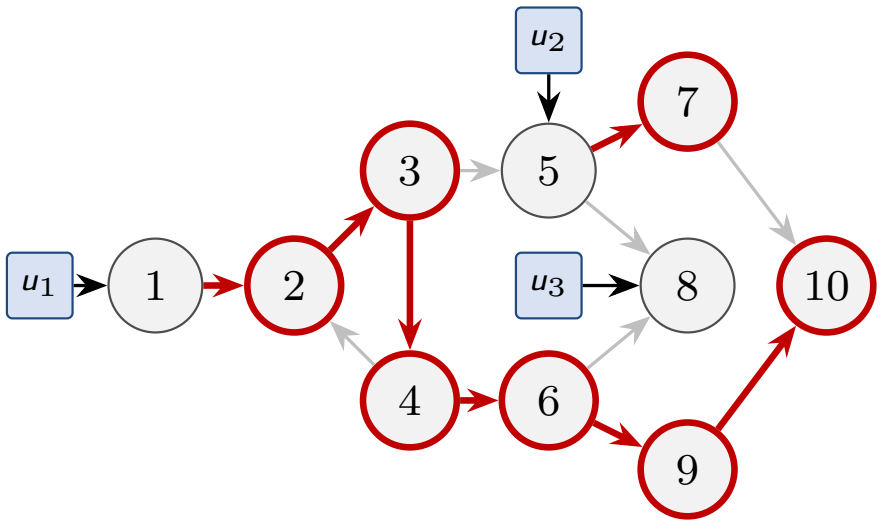


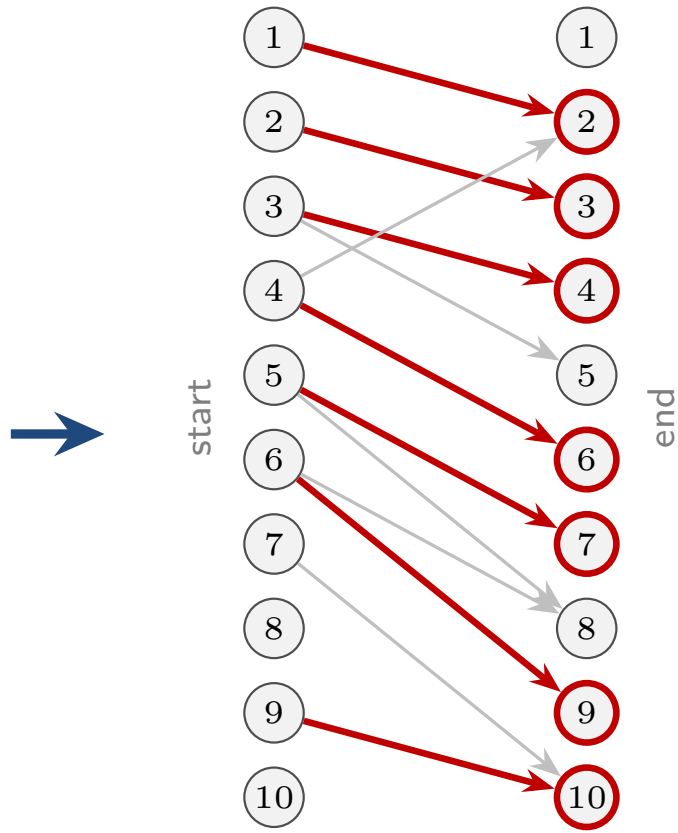


Find a maximum matching first. **Which nodes stay unmatched?**

4 Example: The 10-Node Graph, Solved by Matching

▶ **Step 1: a maximum matching → Step 2: inputs at the unmatched nodes**





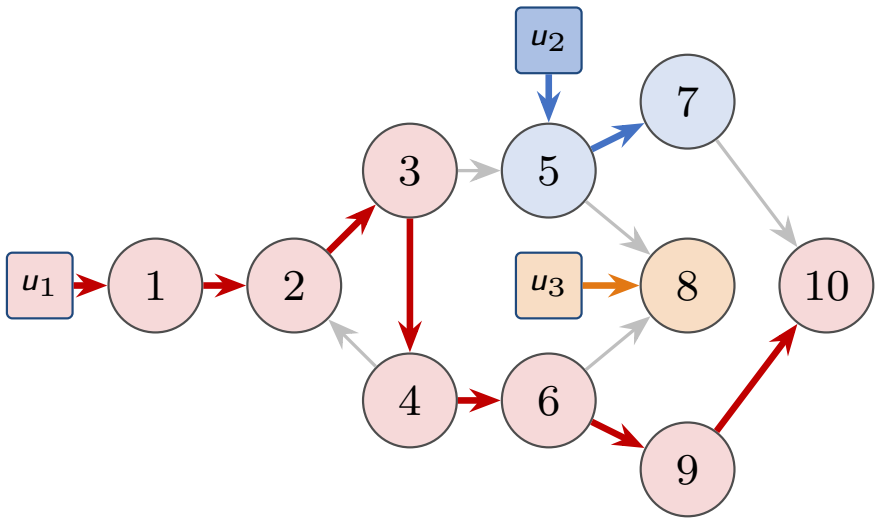
$|M^*| = 7 \rightarrow N_D = 3$ · unmatched 1, 5, 8

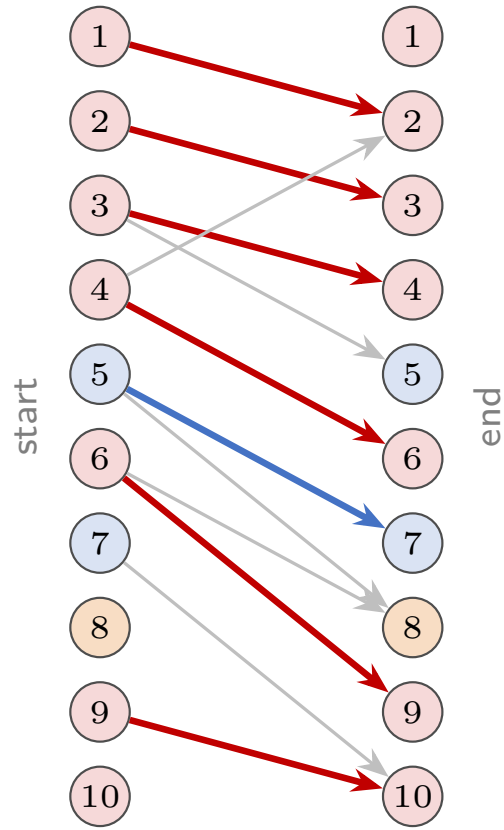
inputs at 1, 5, 8

Nodes 1, 5, 8 stay unmatched, so 3 inputs, not 2. **What do the matching edges form?**

4 Example: The 10-Node Graph, Solved by Matching

▶ **Step 1: a maximum matching → Step 2: inputs at the unmatched nodes**





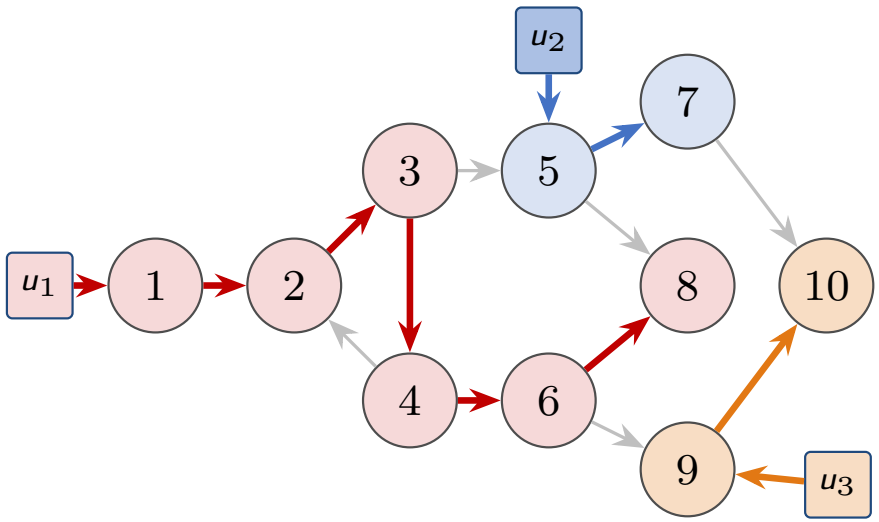
3 stems: **1, 2, 3, 4, 6, 9, 10** · **5, 7** · **8**

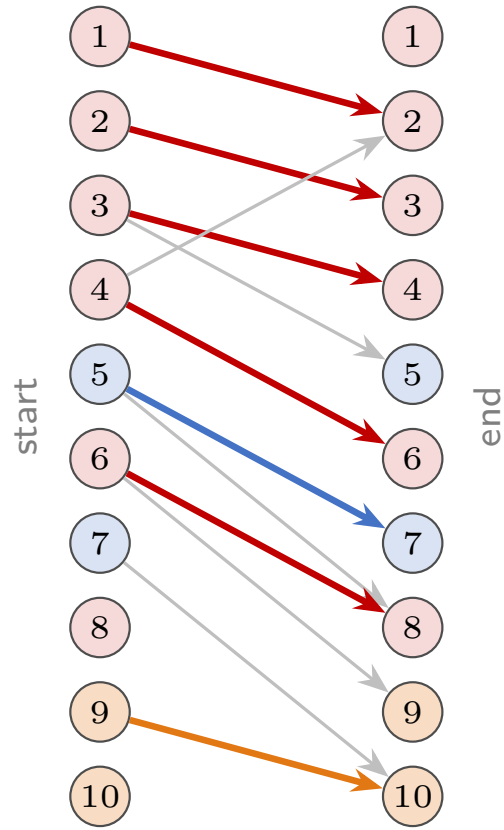
no shared node, all 10 covered → SC

The 3 stems share no node and cover all 10. **Is this the only maximum matching?**

4 Example: The 10-Node Graph, Solved by Matching

▶ **Step 1: a maximum matching → Step 2: inputs at the unmatched nodes**





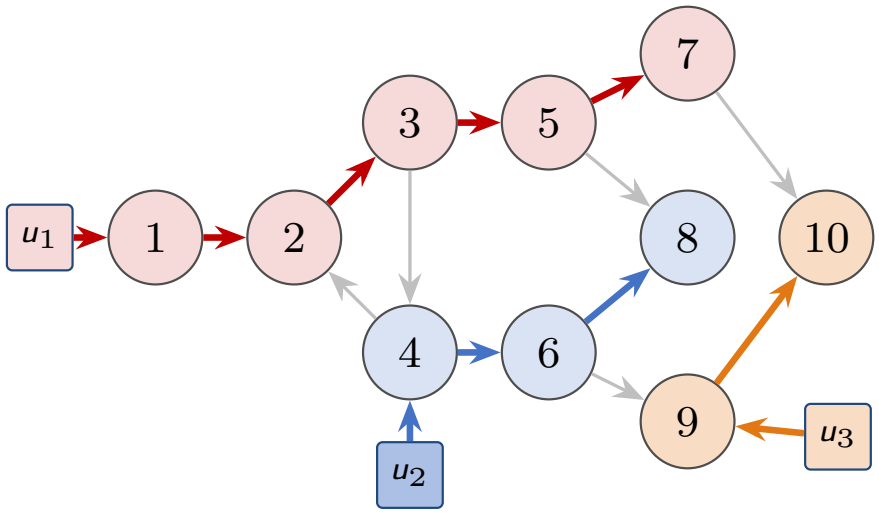
$6 \to 8$ instead of $6 \to 9$ → drivers 1, 5, 9

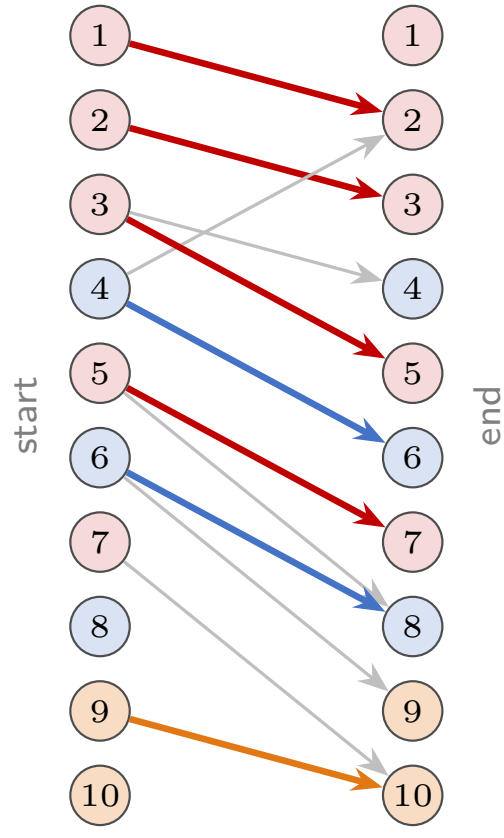
stems: **1, 2, 3, 4, 6, 8** · **5, 7** · **9, 10**

Matching $6 \to 8$ instead of $6 \to 9$ also keeps 7 edges. **Can another edge change too?**

4 Example: The 10-Node Graph, Solved by Matching

▶ **Step 1: a maximum matching → Step 2: inputs at the unmatched nodes**





$3 \to 5$ instead of $3 \to 4$ → drivers 1, 4, 9

stems: **1, 2, 3, 5, 7** · **4, 6, 8** · **9, 10**

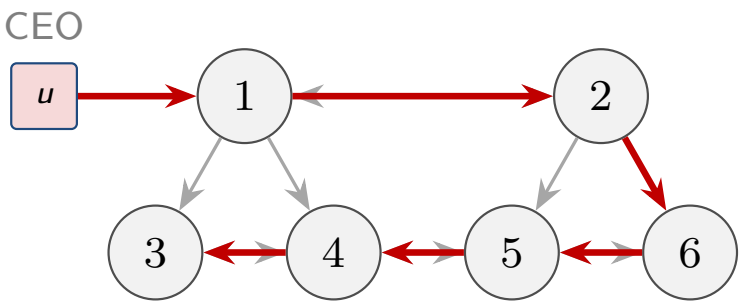
More maximum matchings exist, so more driver sets

$N_D = 3$ stays, but the driver set changes. **What is a driver in a real network?**

5 Application: Social Networks, Whom to Tell First?



- **Goal** $\rightarrow$ every employee backs a new plan
- **Nodes** $\rightarrow$ employees
- **Edges** $\rightarrow$ who persuades whom
- **Drivers** $\rightarrow$ employees the CEO briefs



Links down and across layers

$\rightarrow$ 1 **stem** through all 6

$\rightarrow$ **1 driver**: brief 1 executive

Company social networks: drivers = 1–4% of employees

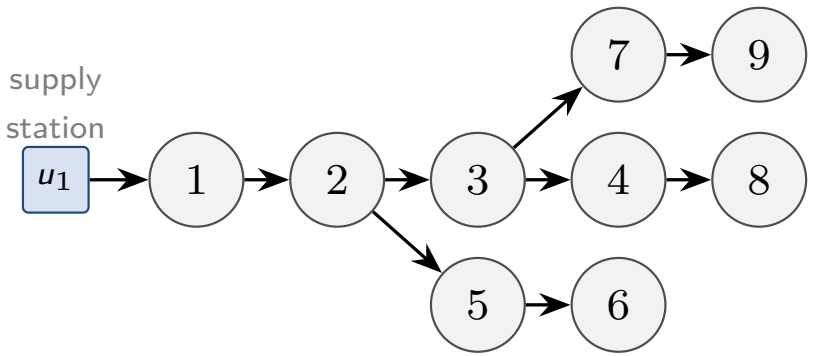
Table 1, p. 169: a 77-person firm needs 1 driver, a 46-person firm 2

A 77-person company needs only 1 driver. **What about North Korea?**

5 Application: Power Grid, Where to Feed Power In?



- **Goal** $\rightarrow$ each house its own amount $\rightarrow$ **SC**
- **Nodes** $\rightarrow$ houses
- **Edges** $\rightarrow$ power lines
- **Drivers** $\rightarrow$ houses fed by a supply station



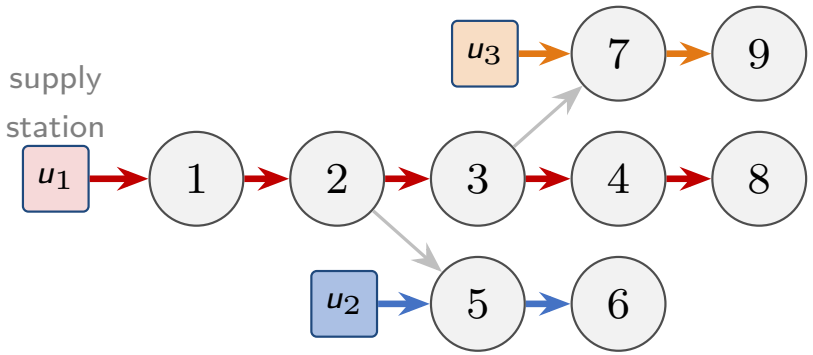
Texas grid (Table 1): 4,889 nodes, only 5,855 power lines $\rightarrow$ drivers 32.5%, about 1 in 3

Power from 1 station reaches all 9 houses. **Can it give each house its own amount?**

5 Application: Power Grid, Where to Feed Power In?



- **Goal** $\rightarrow$ each house its own amount $\rightarrow$ **SC**
- **Nodes** $\rightarrow$ houses
- **Edges** $\rightarrow$ power lines
- **Drivers** $\rightarrow$ houses fed by a supply station



1 station $\rightarrow$ 1 **stem**, only 5 of 9

Split at 2: houses 3 and 5 move together

2 splits $\rightarrow$ 2 more stations $\rightarrow$ $N_D = 3$

4 minimum driver sets: $\{1, 5, 7\}$, $\{1, 3, 7\}$, $\{1, 4, 5\}$, $\{1, 3, 4\}$

Texas grid (Table 1): 4,889 nodes, only 5,855 power lines $\rightarrow$ drivers 32.5%, about 1 in 3

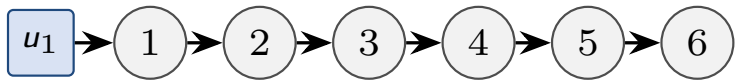
$N_D = 3$ is unique, but its driver set is not. **Which set should we choose?**

6 Conclusion: Where Should a 2nd Input Go?

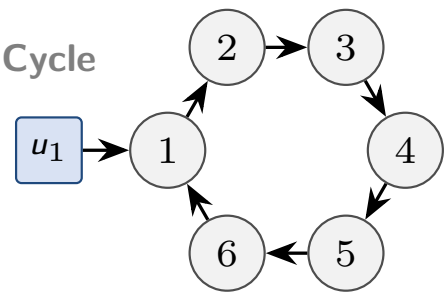
CONCEPT Limitation

- N_D **unique**, minimum driver set **not unique** (star: 3 sets, grid: 4)
- Least **control energy** $\int_0^T \|u\|^2 dt$ $\rightarrow$ which set? $\rightarrow$ not answered

Path



Cycle

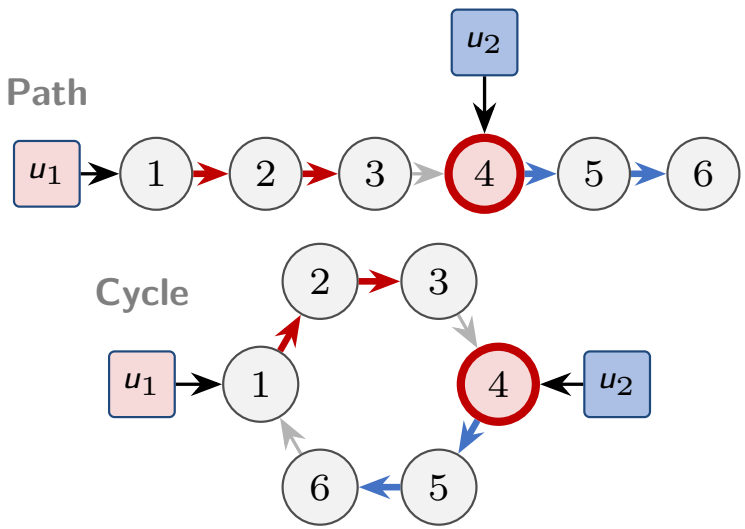


1 input at node 1 already controls both graphs. **Where should a 2nd input go?**

6 Conclusion: Where Should a 2nd Input Go?

CONCEPT Limitation

- N_D **unique**, minimum driver set **not unique** (star: 3 sets, grid: 4)
- Least **control energy** $\int_0^T \|u\|^2 dt$ $\rightarrow$ which set? $\rightarrow$ not answered



Path: **node 4** $\rightarrow$ 2 stems of 3 nodes

Cycle: **node 4**, the opposite node

Both: the longest stem cut in **half**

Energy: node 4 $\approx$ **908**, node 2 or 6 $\approx 3.2 \times 10^7$

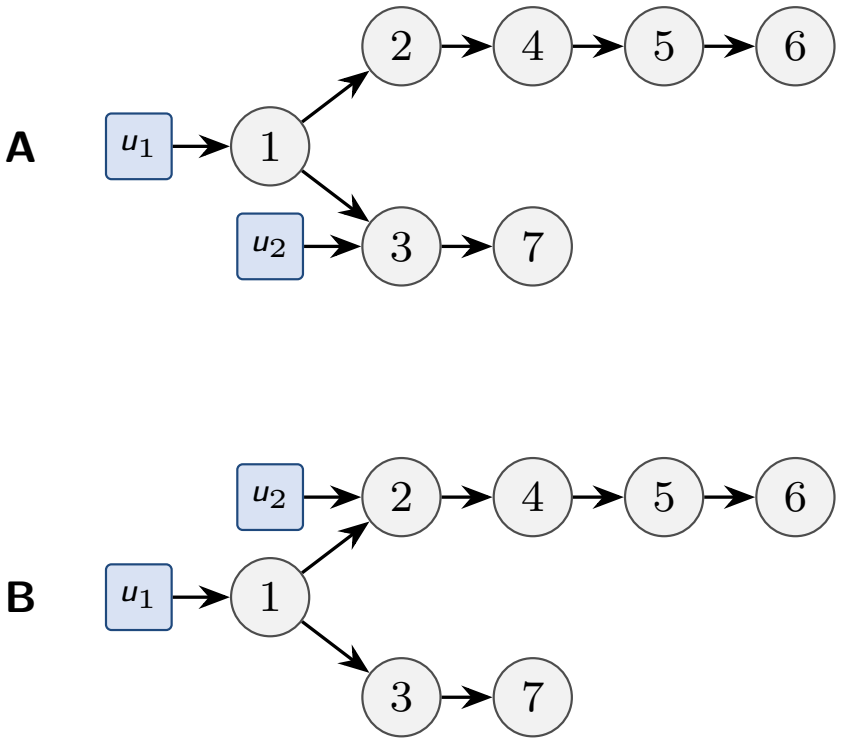
hardest target of length 1, $T = 1$, weights 1

how to compute it: beyond this course

Node 4 gives 2 equal stems. **Does equal splitting work on a branching graph?**

6 Conclusion: Which Minimum Set to Choose?

- **Tree: no cycle, 1 edge into each node except node 1**



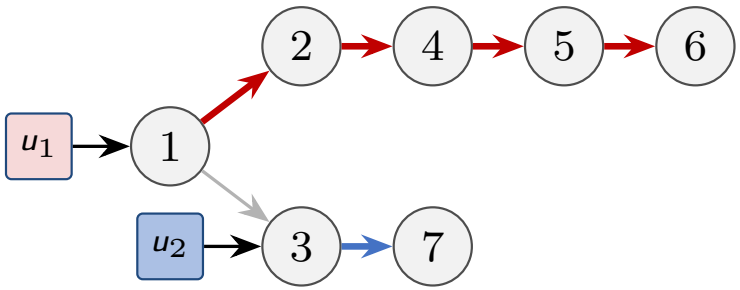
worst-case energy, $T = 1$, weights 1

This tree has only 2 minimum driver sets. **Which set needs less control energy?**

6 Conclusion: Which Minimum Set to Choose?

▶ **Tree: no cycle, 1 edge into each node except node 1**

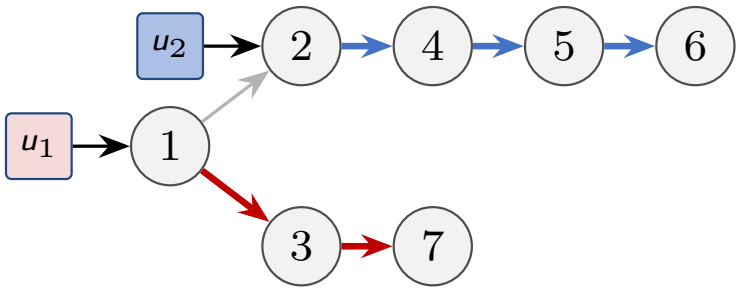
A



A: stems of **5** and 2 nodes

energy $\approx 3.2 \times 10^7$

B



B: stems of 3 and **4** nodes

energy $\approx 1.3 \times 10^5$

worst-case energy, $T = 1$, weights 1

Idea, no proof: shorten the longest stem = balance the stem lengths

Balanced stems needed less energy here. **Does that still hold with weights or cycles?**

Talk Outline: Parts 0 to 6

CONCEPT The Rule

- **Every student must follow your talk** $\rightarrow$ intuition and examples

- **0 Paper Title** $\rightarrow$ title, authors, journal, year
- **1 Motivation** $\rightarrow$ why the main result is needed, and what it solves
- **2 Preliminaries** $\rightarrow$ the concepts it needs, each with a small example
- **3 Main Result** $\rightarrow$ the statement, then its intuitive meaning
- **4 Example** $\rightarrow$ a graph the other students can work through by hand
- **5 Application** $\rightarrow$ where the result can be used
- **6 Conclusion** $\rightarrow$ limitations, and an idea to improve them

By Oct 21, send me your team, paper, and week. **Which weakness will your team find?**